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The word problem for Artin groups of FC type

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Abstract

The Artin groups of FC type can be characterized as the smallest class of Artin groups which is closed under free products amalgamated over special subgroups (subgroups generated by a subset of canonical generators) and which contains the finite-type Artin groups. There is a computationally feasible normal form for special cosets of FC-type Artin groups. In particular, the word problem is solvable in quadratic time. It is also shown that FC-type Artin groups are asynchronously automatic and that the set of positive words in a standard presentation of an Artin group of FC type is isomorphic to the monoid given by the same presentation. © 1998 Elsevier Science B.V. All rights reserved.

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0. Introduction

Artin groups are a natural generalization of braid groups. Braid groups can be thought of as encoding all the patterns that can be woven into a set of (almost) parallel strands. Artin found a presentation for each braid group as a finite system of generators and relations and solved the word problem [1]; i.e., he gave an algorithm which decides whether a given product of generators is the identity element of the group. The braid groups have been well studied since then and are of continuing interest. They have applications to the study of knots and links and are related to the mapping class groups.

The braid groups are contained in a larger class of Artin groups known as the finite-type Artin groups. Many of the properties of braid groups extend naturally to finite-type Artin groups. Garside [10] and Deligne [8] solved the word problem for finite-type Artin groups. Thurston [9] showed that braid groups are biautomatic. Biautomaticity implies, among other things, that the word problem can be solved in quadratic time. Charney [4] showed that the finite-type Artin groups are biautomatic. The Artin groups

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of infinite-type are not amenable to the same techniques and very little is known about them. Peifer has shown that some of the infinite-type Artin groups have solvable word problem, more specifically, that those of large type are automatic [13] and that those of extra-large type are biautomatic [14]. Chermak has shown that locally non-spherical Artin groups have solvable word problem [7]. Van Wyk [17] has shown that the right-angled Artin groups are biautomatic and it follows from work of Hermiller and Meier [11] that, in fact, any graph product of biautomatic Artin groups is a biautomatic Artin group.

What follows is a solution to the word problem for another class of infinite-type Artin groups, Artin groups of FC type. This class is defined by Charney and Davis [6] as follows. Let (W,S) be a Coxeter system. The associated Artin group A is of *FC type* if the following condition is satisfied: if $T \subseteq S$ and every pair of elements of T generates a finite subgroup of W, then T generates a finite subgroup of W. The Artin groups of FC type can be characterized as the smallest class of Artin groups closed under amalgamations over special subgroups and containing the finite-type Artin groups. This class contains graph products of finite-type Artin groups (and hence the right-angled Artin groups) but is essentially different from the locally non-spherical and large type Artin groups. The solution is via a normal form which yields an asynchronously automatic structure. It is not known, in general, whether asynchronously automatic groups admit a polynomial-time solution to the word problem. The normal form presented below has computational properties which are independent of the generic algorithms associated with automaticity. These properties yield a quadratic time solution to the word problem.

Section 1 provides background on Artin groups collected principally from papers of Charney [4, 5] and Charney and Davis [6]. Section 2 gives the core result: Artin groups of finite type have a system of special coset representatives analogous to minimal coset representatives of Coxeter groups. The representatives of the trivial subgroup reconstitute the geodesic normal form in [5]. The coset representatives of those special subgroups containing positive elements coincide with those defined in [6], which were the inspiration for the generalization given here. Section 3 extends the normal form for coset representatives to FC-type Artin groups. In this case, the representatives of the cosets of the trivial subgroup do not give a biautomatic structure but they do yield a normal form which can be computed in quadratic time. The normal form is used to show the following two facts: (a) the monoid defined by the same generators and relations as in the standard presentation for an FC-type Artin group is exactly the monoid of positive elements of that group and (b) FC Artin groups are asynchronously automatic.

1. Finite-type Artin groups

This section contains preliminary information about finite-type Artin groups due primarily to Garside [10] and Deligne [8]. It uses facts about Coxeter groups that can be found in the first two chapters of [3]. For more details, see [4–6].

A *Coxeter matrix* is a symmetric matrix (m_{ij}) with entries in $\{1,2,\ldots,\infty\}$ such that $m_{ii}=1$ and $m_{ij}\geq 2$, for $i\neq j$. A *Coxeter system* associated to an $n\times n$ Coxeter matrix is a pair (W,S), where $S=\{s_1,s_2,\ldots,s_n\}$ is a finite set and W is the group with presentation

$$W=\langle S\mid (s_is_j)^{m_{ij}}=1,\ m_{ij}\neq\infty\rangle.$$

The corresponding *Artin group* A is the group with presentation

$$A=\langle S\mid \mathrm{prod}(s_i,s_j;m_{ij})=\mathrm{prod}(s_j,s_i;m_{ij})\rangle,$$

where $\mathrm{prod}(s,t;m)$ denotes the alternating product $sts\cdots$ containing m factors. If W is finite, A is said to be of *finite type*. For example, the braid group on n strands, B_n, is an Artin group whose standard presentation is

$$B_n=\langle s_1,\ldots,s_{n-1}\mid s_is_{i+1}s_i=s_{i+1}s_is_{i+1},\ s_is_j=s_js_i \text{ if } |i-j|>1\rangle.$$

Braid groups are of finite type since the Coxeter group W corresponding to B_n is S_n, the symmetric group on n objects. Note that in the presence of the relations $s_i^2=1$, the Artin group relations are equivalent to the corresponding Coxeter group relations. In other words, there is a homomorphism $\pi:A\to W$ given by adding the relations $s_i^2=1$.

For $T\subseteq S$, let (W_T,T) denote the Coxeter system corresponding to the $T\times T$ submatrix of the Coxeter matrix for W. Let A_T denote the Artin group corresponding to W_T. It follows from Tits' solution to the word problem in Coxeter groups that W_T is isomorphic to the subgroup of W generated by T under the natural map. This is also true for Artin groups; i.e., A_T is isomorphic to the subgroup of A generated by T under the natural map. This was shown by Deligne [8] for A of finite-type and by van der Lek [16] in general. The groups W_T and A_T are called the *special subgroups* of W and A, respectively. Cosets of special subgroups are called *special cosets*. Special subgroups satisfy a nice intersection property [16]: $A_T\cap A_U=A_{T\cap U}$.

For the rest of this section, suppose W is finite and A is the associated finite-type Artin group. Let A^+ be the monoid with the same presentation as A; i.e., $A^+=F(S)^+/\sim$, where $F(S)^+$ is the free monoid on S and $\sim$ is the equivalence relation generated (via transitive closure) by the equivalences $uwv\sim uw'v$ if $w=\mathrm{prod}(s,t;m_{st})$ and $w'=\mathrm{prod}(t,s;m_{st})$, for some $s,t\in S$. Note that since the monoid relations preserve length, word length is additive in A^+. Deligne [8, Theorem 4.14] has shown that the natural map $A^+\to A$ is an injection. Thus, A^+ may be regarded as a submonoid of A, called the monoid of *positive elements*.

To each element w of the Coxeter group W, there is a unique positive element $\mu\in A^+$ of minimal length such that $\pi(\mu)=w$. If $\mu\neq 1$, we call μ a *minimal*. The set of minimals is denoted by M. Since $S\subseteq M$, M is another finite generating set for A. The word length of an element $a\in A$ over the generating set M will be called the M-length of A, denoted $|a|_M$; similarly, the word length over S will be called S-length, denoted $|a|_S$. The generating set M is easier to work with for some purposes.

For example, Paterson and Razborov [12] have shown that unless P = NP, there is no polynomial-time algorithm independent of the number of strands to produce a minimal length representation of a braid from a given one in the generators S. However, there is such an algorithm over the generating set M [9, Corollary 9.5.3]. For $T \subseteq S$, let M_T be the set of minimals in A_T.

Lemma 1 (Charney [4, Lemma 2.3]). *Let Δ be the minimal corresponding to the (unique) longest element of W. For all $a \in A$, $\mu \in M$, and $s \in S$,*

(i) $\Delta^2 a = a\Delta^2$,
(ii) *there is* $\mu^* \in M$ *such that* $\Delta = \mu^* \mu$,
(iii) *there is* $\bar{\mu} \in M$ *such that* $\Delta\mu = \bar{\mu}\Delta$, *and*
(iv) *there is* $\bar{s} \in S$ *such that* $\Delta s = \bar{s}\Delta$.

For $T \subseteq S$, let Δ_T denote the minimal corresponding to the longest element of W_T. Define the partial orderings $\preceq_\ell$ and $\preceq_r$ on A^+ by

$$a \preceq_r b \text{ if } b = ca \text{ for some } c \in A^+$$

and

$$a \preceq_\ell b \text{ if } b = ac \text{ for some } c \in A^+.$$

A *lattice* is a partially ordered set (poset) in which each pair of elements has both an infimum and a supremum. The infimum of x and y is denoted by $x \wedge y$ and is called the *meet* of x and y; the supremum is denoted by $x \vee y$ and is called the *join.* Charney and Davis [6, Lemma 4.5.2] show that the posets $(A^+, \preceq_\ell)$ and $(A^+, \preceq_r)$ are lattices. The operations in these lattices will be denoted $\wedge_\ell$, $\vee_\ell$ and $\wedge_r$, $\vee_r$, respectively. For $a, b \in A^+$, the statement $a \wedge_* b = 1$ will sometimes be abbreviated to $a \perp_* b$, for $* = \ell$ or r. For any $T \subseteq S$ and any $p \in A^+$, the following statements are equivalent: (a) $p \in M_T$; (b) $p \preceq_\ell \Delta_T$; (c) $p \preceq_r \Delta_T$.

For any $a \in A^+$, the sets

$$\{\mu \in M \cup \{1\} : \mu \preceq_\ell a\}$$

and

$$\{\mu \in M \cup \{1\} : \mu \preceq_r a\}$$

have unique maximal elements denoted $\mathrm{maxmin}_\ell(a)$ and $\mathrm{maxmin}_r(a)$, respectively, [8]. We can put any $p \in A^+$ into a normal form over M by letting $p_1 = \mathrm{maxmin}_r(p)$, $p_2 = \mathrm{maxmin}_r(pp_1^{-1})$, $p_3 = \mathrm{maxmin}_r(pp_1^{-1}p_2^{-1})$, and so on until $p_{k+1} = 1$ and writing $p = p_k \cdots p_2 p_1$. This normal form for positive elements is called the *right greedy canonical minimal decomposition* (*rmd*). A left normal form (lmd) can be defined similarly and symmetric versions of the following properties of rmds hold for lmds. The following lemma is a fundamental rmd property [4, Lemma 2.4].

Lemma 2. *For* $a, b \in A^+$, $\mathrm{maxmin}_r(ab) = \mathrm{maxmin}_r(\mathrm{maxmin}_r(a)b)$.

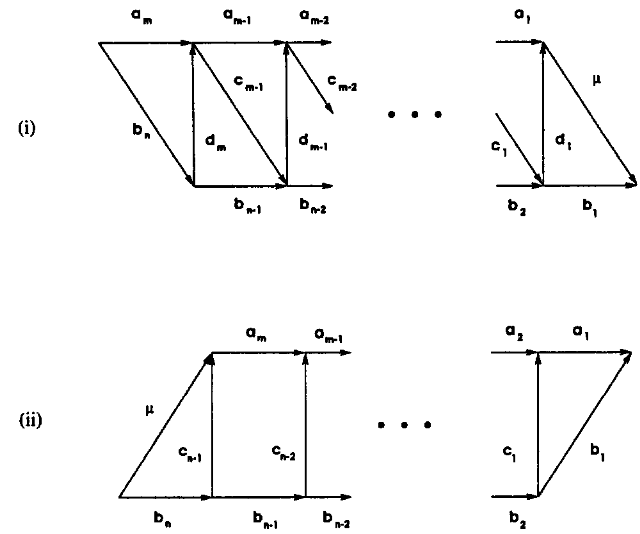

Fig. 1. Multiplying rmds by a minimal.

It follows from Lemma 2 that (a) a word $p_k \cdots p_2 p_1$ over M is an rmd if and only if $p_i = \text{maxmin}_r(p_{i+1} p_i)$ for $i = 1, 2, \ldots, k-1$ and (b) the language of rmds is a regular language over M.

The following lemmas describe what happens to orthogonal elements [5, Lemma 2.8] and rmds [5, Lemma 3.1] after multiplication by a single minimal (Fig. 1).

Lemma 3. *Let* $a, b \in A^+$, $\sigma \in M$. *If* $a \perp_r b$ *then* $\sigma a \wedge_r b \in M \cup \{1\}$.

Lemma 4. *Let* $a, b \in A^+$ *have rmds* $a = a_m a_{m-1} \cdots a_1$ *and* $b = b_n b_{n-1} \cdots b_1$ *and let* $\mu \in M$. *Then*

(i) *if* $b = a\mu$ *then* $n = m$ *or* $m+1$ *and there are* $c_i, d_i \in M \cup \{1\}$ *such that* $a_i = c_i d_i$ *and* $b_i = d_i c_{i-1}$, *where* $d_{m+1} = 1$ *and* $c_0 = \mu$; *and*

(ii) *if* $b = \mu a$ *then* $n = m$ *or* $m+1$ *and there are* $c_i \in M \cup \{1\}$ *such that* $b_n b_{n-1} \cdots b_i c_{i-1} = \mu a_m \cdots a_i$.

It follows that if $a \preceq_* b$ then $|a|_M \leq |b|_M$, for $* = r$ or ℓ. By Lemma 1, it is possible to write each $a \in A$ as $a = p\Delta^{-k}$ for some $p \in A^+$ and some $k \geq 0$. After cancelling as much as possible between the positive and negative parts, the form ab^{-1} is obtained with $a, b \in A^+$ such that $a \perp_r b$. This form will be called the *right normal form* or *rnf*. If a and b are written as rmds, the result is a normal form over $M \cup M^{-1}$ for elements of A. This normal form will also be called rnf when it is clear that we are considering words rather than groups elements. Charney [5] has shown that the language of rnfs is a biautomatic geodesic normal form with uniqueness.

Let $T \subseteq S$. For $n \geq 1$, $\Delta_T = \text{maxmin}_r \Delta_T^n$ and so Δ_T^n is an rmd. Also, every positive element of A_T preceeds Δ_T^n for sufficiently large n under both partial orderings $\preceq_\ell$ and $\preceq_r$. Consequently, certain expressions involving powers of Δ_T become stable for sufficiently large powers. For example, in Lemma 6 it is shown that if $T \subseteq S$, $a, b \in A^+$, and $E(n)$ denotes the expression $a \wedge_r \Delta_T^n b$, then $E(n) = E(|a|_M)$ for all $n \geq |a|_M$.

Definition 1. Suppose that for each integer n, $E(n)$ is an expression over elements of A, products, powers, and lattice operations. Suppose there is an integer N such that for all $n \geq N$, $E(n) = E(N)$. Then the set $\{n \in \mathbf{N} : n \geq N\}$ is called the *stable range* of $E(n)$.

The following lemmas give bounds for the stable ranges of some eventually stable expressions.

Lemma 5. *Let $T \subseteq S$. Let $b, c \in A^+$ with $k = |c|_M$. If $\Delta_T^n b \succeq_r c$ for some $n \in \mathbf{N}$, then $\Delta_T^k b \succeq_r c$.*

Proof. This is clearly true if $k \geq n$ so suppose otherwise. Let $c = c_k c_{k-1} \cdots c_1$ (rmd). The proof is by induction on k. For $k = 1$, $c = c_1 \in M$ implies

$$c_1 \preceq_r \text{maxmin}_r(\Delta_T^n b) = \text{maxmin}_r(\text{maxmin}_r(\Delta_T^n)b) = \text{maxmin}_r(\Delta_T b).$$

Thus, $c \preceq_r \Delta_T b$. For $k > 1$, $c \preceq_r \Delta_T^n b$ implies $c_{k-1} \cdots c_1 \preceq_r \Delta_T^n b$ so $\Delta_T^{k-1} b \succeq_r c_{k-1} \cdots c_1$ by induction. Thus, $\Delta_T^{k-1} b = b_1 c_{k-1} \cdots c_1$, for some $b_1 \in A^+$. Since $c \preceq_r \Delta_T^n b$, $\Delta_T^n b = b_2 c$, for some $b_2 \in A^+$. Thus,

$$b_2 c_k c_{k-1} \cdots c_1 = \Delta_T^n b = \Delta_T^{n-k+1} \Delta_T^{k-1} b = \Delta_T^{n-k} \Delta_T b_1 c_{k-1} \cdots c_1.$$

Cancelling $c_{k-1} \cdots c_1$ gives $c_k \preceq_r \Delta_T^{n-k+1} b_1$. As in the $k = 1$ case, this implies $c_k \preceq_r \Delta_T b_1$. Thus, for some $b_3 \in A^+$,

$$\Delta_T^k b = \Delta_T \Delta_T^{k-1} b = \Delta_T b_1 c_{k-1} \cdots c_1 = b_3 c_k c_{k-1} \cdots c_1. \qquad \square$$

Lemma 6. *Let $T \subseteq S$ and $a, b \in A^+$. Then $|a|_M$ is in the stable range of $a \wedge_r \Delta_T^n b$.*

Proof. Let $c = a \wedge_r \Delta_T^n b$. Then $c \preceq_r a$ so $|c|_M \leq |a|_M$. By Lemma 5, $c \preceq_r \Delta_T^{|c|_M} b \preceq_r \Delta_T^{|a|_M} b$. Hence, $c \preceq_r a \wedge \Delta_T^{|a|_M} b$. On the other hand, if $n \geq |a|_M$ then $a \wedge \Delta_T^{|a|_M} b \preceq_r c$. $\square$

The notation and terminology of [9] for dealing with normal forms will be used when it is necessary to more carefully distinguish between groups elements and words over a generating set. For any finite generating set X of a group G, let X^{-1} denote a set disjoint from X of formal inverses of the elements of X. Let $X^{\pm 1} = X \cup X^{-1}$. Let $X^{\pm *} = (X \cup X^{-1})^*$ denote the free monoid of finite sequences or *words* over $X^{\pm 1}$. If $Y \subseteq X$, $Y^{\pm *}$ is a submonoid of $X^{\pm *}$. The sequence of length zero is called the *empty word* and is denoted by ε. There is a natural monoid epimorphism from $X^{\pm *}$

to G. If w is a word in $X^{\pm *}$, let $\overline{w}$ denote the image of w under this natural monoid epimorphism.

For $a, b \in M^*$, say $a \perp_r b$ if $\overline{a} \perp_r \overline{b}$ and let $a \wedge_r b$ be the word corresponding to the rmd of $\overline{a} \wedge_r \overline{b}$. Extend $\perp_\ell$ and $\wedge_\ell$ similarly. These lattice operations on words can be computed in quadratic time.

Lemma 7. *For $a, b \in M^*$, the lattice operations $a \wedge_\ell b$ and $a \wedge_r b$ can be computed in $O((|a|_M + |b|_M)^2)$ time.*

Proof. It suffices to show this for $\wedge_r$ since the argument for $\wedge_\ell$ is symmetric. Let $a = a_m a_{m-1} \cdots a_1$. By Lemma 2, $\operatorname{maxmin}_r(a)$ can be found by reading a from left to right in pairs: $a_m a_{m-1}$ is replaced by $a'_m a'_{m-1}$ (rmd), $a'_{m-1} a_{m-2}$ is replaced by $a''_{m-1} a'_{m-2}$ (rmd), and so on, where the first factor in a replacement pair may be 1. Then $\operatorname{maxmin}_r a = a'_1$. Similarly compute $\operatorname{maxmin}_r b$. Set $c_1 = \operatorname{maxmin}_r a \wedge_r \operatorname{maxmin}_r b$. (There are finitely many possible meets of minimals which may be considered to have been computed in advance.) Compute $\operatorname{maxmin}_r(ac_1^{-1})$ by replacing a'_1 by $a'_1 c_1^{-1}$ in the string $a''_m \cdots a''_2 a'_1$ and processing the string from left to right as before. Compute $\operatorname{maxmin}_r(bc_1^{-1})$ similarly and set $c_2 = \operatorname{maxmin}_r(ac_1^{-1}) \wedge_r \operatorname{maxmin}_r(bc_1^{-1})$. Repeat until $c_{k+1} = 1$. Then $a \wedge_r b = c_k c_{k-1} \cdots c_1$, with $k \le \min(|a|_M, |b|_M)$. Thus, $a \wedge_r b$ is calculated in $2(k+1)$ passes of length at most $\max(|a|_M, |b|_M)$. □

The following lemma provides nice coset representatives for special cosets containing positive elements [6, Lemma 4.5.3]. It will be generalized in the following section.

Lemma 8. *Let $a \in A$ and $T \subseteq S$. If $aA_T \cap A^+ \neq \emptyset$ then it contains a least element with respect to $\preceq_\ell$.*

2. Finite-type minimal coset representatives

Any Artin group of FC type can be written as a free product with amalgamation whose factors are (ultimately) finite-type Artin groups. The standard normal form for elements of an amalgamated product requires sets of coset representatives in the factor groups. This section describes a good set of coset representatives for finite-type Artin groups. (The amalgam normal form is described in the next section.)

Let (W, S) be a Coxeter system with W finite and let A be the corresponding (finite-type) Artin group. Each special coset of W has a unique coset representative of minimal length over S (see [3]). There is an analogous system of distinguished coset representatives for A. However, length over the generating set M does not provide unique special coset representatives for A. A stronger partial ordering on A is needed. Such a partial ordering is defined below as a lexicographic combination of the partial orderings on A^+ discussed in the previous chapter. Its main properties are summarized in the following theorem.

Theorem 1. *There is a partial ordering $\preceq_R$ on A such that*
(i) *every special coset xA_T has a least element $m(xA_T)$ with respect to $\preceq_R$,*
(ii) *$m(xA_T)$ has minimal M-length among coset representatives, and*
(iii) *if A_U is a special subgroup such that $A_U \cap xA_T \neq \emptyset$ then $m(A_U \cap xA_T) = m(xA_T)$.*

It follows from van der Lek's intersection property ($A_U \cap A_T = A_{U\cap T}$) that if $A_U \cap xA_T$ is nonempty then $A_U \cap xA_T$ is a coset of the special subgroup $A_{U\cap T}$. Thus, assuming property (i), the expression $m(A_U \cap xA_T)$ in property (iii) is well-defined. Property (iii) implies that $m(xA_T)$ lies in every nonempty intersection of the form $A_U \cap xA_T$ and hence can be written in terms of the smallest subset of canonical generators sufficient to express some element of xA_T.

Let $x \in A$ and $T \subseteq S$. Let $x = ab^{-1}$ (rnf). Define $m_T(x)$ by

$$m_T(x) = x\Delta_T^{-m}(\Delta_T^m b(a \wedge_r \Delta_T^m b)^{-1} \wedge_\ell \Delta_T^n),$$

where m is in the stable range of $E(i) = a \wedge_r \Delta_T^i b$ and n is in the stable range of $F(j) = b(a \wedge_r \Delta_T^m b)^{-1} \wedge_\ell \Delta_T^j$. This definition can be understood as an algorithm for finding a distinguished representative $m_T(x)$ for the coset xA_T. This algorithm is outlined by the following system of equations, which also yields $m_T(x)$:

$$g = a \wedge_r \Delta_T^m b, \tag{1}$$

$$a_1 = ag^{-1}, \tag{2}$$

$$b_1 = \Delta_T^m bg^{-1}, \tag{3}$$

$$h = b_1 \wedge_\ell \Delta_T^n, \tag{4}$$

$$b_2 = h^{-1}b_1, \tag{5}$$

$$m_T(x) = a_1 b_2^{-1}, \tag{6}$$

where m is in the stable range of g and n is in the stable range of h. For example, consider the element $x = \Delta_T s_3^{-1}$ of the braid group B_4, where $T = \{s_1, s_2\}$. As a product of minimals, $\Delta_T s_3^{-1}$ is in rnf; no cancellation can occur between the positive and negative part. The goal is to find a representative of the coset xA_T whose positive part is as "small" as possible. Multiplication on the right by Δ_T^{-1} induces some cancellation:

$$x\Delta_T^{-1} = s_1s_2s_1s_3^{-1}s_1^{-1}s_2^{-1}s_1^{-1} = s_1s_2s_1s_1^{-1}s_3^{-1}s_2^{-1}s_1^{-1} = (s_1s_2)(s_1s_2s_3)^{-1}.$$

However, another factor of Δ_T^{-1} yields no further cancellation; $m = 1$ is in the stable range of g. We have $a_1 = s_1s_2$ and $b_1 = s_1s_2s_3$. Having reduced the positive part, we let A_T reabsorb as much of the negative part as possible: $h = s_1s_2$, $b_2 = s_3^{-1}$, and $m_T(x) = s_1s_2s_3^{-1}$.

Remark 1. By Lemma 6, $m = |a|_M$ is in the stable range of g. Thus, $|b_1|_M \leq |a|_M + |b|_M$ and $n = |a|_M + |b|_M$ is in the stable range of h. However, an efficient calculation could use the minimal number of Δ_Ts (plus one to check the termination condition) by taking advantage of the fact that $maxmin_r(\Delta_T^n b) = maxmin_r(\Delta_T b)$. For example, to compute g, proceed as in the proof of Lemma 7 to find $c_1 = maxmin_r(a \wedge_r \Delta_T b)$. In the next pass, use $\Delta_T b'$ in place of b', the replacement string for $\Delta_T b$. Continue by multiplying by one Δ_T at a time.

Remark 2. Note that $m_T(x) \in xA_T$ since by definition, $m_T(x) = xk$, where $k \preceq_\ell \Delta_T^n$.

Remark 3. The elements a_1, b_1, and hence $m_T(x)$ do not depend on the orthogonality of a and b since any common right tail will be absorbed by g. Thus, $m_T(x)$ can be calculated from any pair $a, b \in A^+$ such that $x = ab^{-1}$.

In fact, $m_T(x)$ does not really depend on x but only on the coset xA_T.

Lemma 9. *If $y \in xA_T$ then $m_T(y) = m_T(x)$.*

Proof. Let $y \in xA_T$. Then $y = xw$ for some $w \in A_T$. Write $w = e\Delta_T^{-\ell}$ and $x = c\Delta^{-k}$, where $e \in A_T^+$, $c \in A^+$, and k and ℓ are even. Then $y = ce\Delta^{-k}\Delta_T^{-\ell}$. By Remark 3, $m_T(x)$ can be computed by letting $a = c$ and $b = \Delta^{-k}$ in Eqs. (1)–(6):

$$g = c \wedge_r \Delta_T^m \Delta^k, \tag{7}$$

$$a_1 = cg^{-1}, \tag{8}$$

$$b_1 = \Delta_T^m \Delta^k g^{-1}, \tag{9}$$

$$h = b_1 \wedge_\ell \Delta_T^n, \tag{10}$$

$$b_2 = h^{-1} b_1, \tag{11}$$

$$m_T(x) = a_1 b_2^{-1}, \tag{12}$$

where m is even and into the stable range of $c \wedge_r \Delta_T^m \Delta^k$ by at least the length of e. Similarly, $m_T(y)$ can be computed by letting $a' = ce$ and $b' = \Delta_T^\ell \Delta^k$:

$$g' = ce \wedge_r \Delta_T^m \Delta_T^\ell \Delta^k \tag{13}$$

$$= (c \wedge_r \Delta_T^m \Delta_T^\ell \Delta^k e^{-1})e \tag{14}$$

$$= (c \wedge_r e^* \Delta_T^{m-i} \Delta_T^\ell \Delta^k)e \tag{15}$$

$$= (c \wedge_r \Delta_T^{m-i} \Delta_T^\ell \Delta^k)e \tag{16}$$

$$= ge, \tag{17}$$

where i is the length of e and $*$ is with respect to Δ_T. In going from (15) to (16) the required fact is that since $e^* \in A_T^+$, any bounded (by c in this case) right tail of $e^* \Delta_T^M \Delta^k$

is also a bounded right tail of $\Delta_T^M \Delta^k$ for sufficiently large M while the converse is obvious.

$$a_1' = a'g'^{-1} = ce(ge)^{-1} = cg^{-1} = a_1, \tag{18}$$

$$b_1' = \Delta_T^m b' g'^{-1} \tag{19}$$

$$= \Delta_T^m \Delta_T^\ell \Delta^k (ge)^{-1} \tag{20}$$

$$= \Delta_T^m \Delta_T^\ell e^{-1} \Delta^k g^{-1} \tag{21}$$

$$= \Delta_T^m w^{-1} \Delta^k g^{-1} \tag{22}$$

$$= w^{-1} \Delta_T^m \Delta^k g^{-1} \tag{23}$$

$$= w^{-1} b_1, \tag{24}$$

$$h' = b_1' \wedge_\ell \Delta_T^{n'} \tag{25}$$

$$= w^{-1} b_1 \wedge_\ell \Delta_T^{n'} \tag{26}$$

$$= w^{-1}(b_1 \wedge_\ell w \Delta_T^{n'}) \tag{27}$$

$$= w^{-1}(b_1 \wedge_\ell \Delta_T^{n'} w) \tag{28}$$

$$= w^{-1}(b_1 \wedge_\ell \Delta_T^{n'}) \tag{29}$$

$$= w^{-1} h, \tag{30}$$

$$b_2' = h'^{-1} b_1' = (w^{-1}h)^{-1} w^{-1} b_1 \tag{31}$$

$$= h^{-1} b_1 = b_2 \tag{32}$$

$$m_T(y) = a_1' b_2'^{-1} = a_1 b_2^{-1} = m_T(x). \tag{33}$$

□

Remark 4. Note that the above proof shows that a_1 and b_2 themselves only depend on the coset xA_T. In fact, since $a_1 \preceq_\ell a$ for any $a \in A^+$ such that $m_T(x) = ab^{-1}$ for some $b \in A^+$, $a_1 b_2^{-1}$ is the rnf of $m_T(x)$.

By Remark 2 and Lemma 9, there is a way of choosing a distinguished representative $m(xA_T)$ from each special coset xA_T, namely, $m(xA_T) = m_T(x)$. Each distinguished representative is the least element in its coset with respect to the partial ordering on A described below.

Let $(X, \prec)$ be a poset, $Y \subseteq X$. An element $y_0 \in Y$ is *minimal* in Y if for every $y \in Y$, $y \prec y_0$ implies $y = y_0$. If Y contains an element y_0 such that $y_0 \prec y$ for all $y \in Y$, say that Y has a *least* element and call y_0 the *least* element of Y. Least elements are unique by antisymmetry. Least elements are minimal but minimal elements are not always least. However, if Y has a least element y_0, then y_0 is the unique minimal element of Y.

Define the partial ordering $\preceq$ on $(A^+ \times A^+)$ lexicographically from the posets $(A^+, \preceq_\ell)$ and $(A^+, \preceq_r)$; i.e., $(a_1, b_1) \preceq (a_2, b_2)$ if and only if ($a_1 \preceq_\ell a_2$ and $a_1 \neq a_2$) or ($a_1 = a_2$ and $b_1 \preceq_r b_2$). Since each element x of A has a unique rnf, this induces a partial ordering $\preceq_R$ on A. Define $\rho : A \to A^+ \times A^+$ by $\rho(x) = (a, b)$, where $x = ab^{-1}$ (rnf). Then $\rho(A) = \{(a, b) \in A^+ \times A^+ : a \perp_r b\}$ and $\rho^{-1} : \rho(A) \to A$ is given by $\rho^{-1}(a, b) = ab^{-1}$. For $x, y \in A$, say that $x \preceq_R y$ if $\rho(x) \preceq \rho(y)$. Let $x \in A$ and $T \subseteq S$. Let $x = ab^{-1}$ (rnf). Let $P = \{p \in A^+ : \exists q \in A^+,\ (p, q) \in \rho(xA_T)\}$. For each $p \in P$, let $Q_p = \{q \in A^+ : (p, q) \in \rho(xA_T)\}$. Then for each $p \in P$, $Q_p \neq \emptyset$. Using the notation of Eqs. (1)–(6), it follows from Remark 4 and the definition of a_1 that for any $y \in xA_T$, if $y = pq^{-1}$ with $p, q \in A^+$ then $a_1 \preceq_\ell p$. Thus, P has least element a_1 with respect to $\preceq_\ell$. Similarly, b_2 is the least element with respect to $\preceq_r$ of Q_{a_1}. It follows that $m(xA_T) = a_1 b_2^{-1}$ is the least element with respect to $\preceq_R$ of $xA_T = \rho^{-1}(\bigcup_{p \in P} \{p\} \times Q_p)$. This completes the proof of part (i) of Theorem 1.

To prove part (ii) of Theorem 1, it suffices to show that for any $y \in xA_T$, if $y = ab^{-1}$ with $a, b \in A^+$, then, using the notation of Eqs. (1)–(6), $|a_1|_M \leq |a|_M$ and $|b_2|_M \leq |b|_M$. That $|a_1|_M \leq |a|_M$ is clear from the definition of a_1. The following lemmas are used to prove the second inequality.

Lemma 10. *Suppose $T \subseteq S$, $c \in A^+$, $c = \gamma_1 \gamma_2 \cdots \gamma_k$ (rmd), and $n \geq 0$. Then $\Delta_T^n c = \delta_1 \delta_2 \cdots \delta_m \gamma_1' \gamma_2' \cdots \gamma_k'$ (rmd) for some $\gamma_1', \gamma_2', \ldots, \gamma_k' \in M$ and $\delta_1, \delta_2, \ldots, \delta_m \preceq_r \Delta_T$ with $0 \leq m \leq n$.*

Proof. Lemma 4(ii) and induction. $\square$

Lemma 11. *Let $T \subseteq S$. For any $a \in A^+$ with rmd $a = \alpha_n \cdots \alpha_2 \alpha_1$, let $k(a)$ denote the smallest integer such that $\alpha_j \preceq_r \Delta_T$ for $k(a) < j \leq n$. Then for all $b, g \in A^+$, $k(b) \leq k(bg)$.*

Proof. This proof is by induction on $|g|_M$. For $|g|_M = 0$, $b = bg$ so $k(b) = k(bg)$. Suppose the lemma is true for $|g|_M < m$. Let $b, g \in A^+$, with $g = \gamma_m \cdots \gamma_2 \gamma_1$ (rmd). Let $b' = b\gamma_m \cdots \gamma_2$ and suppose that the rmd of b' is $b' = \beta_n \cdots \beta_2 \beta_1$. Then by Lemma 4(i), there are elements $\sigma_0, \sigma_1, \ldots, \sigma_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ of $M \cup \{1\}$ such that $\beta_i = \sigma_i \alpha_i$ and the rmd of $b'\gamma_1$ is $\eta_j \cdots \eta_2 \eta_1$ with $j = n$ or $j = n + 1$ and $\eta_i = \alpha_i \sigma_{i-1}$ (where $\alpha_{n+1} = 1$ and $\sigma_0 = \gamma_1$). Let $k = k(b')$. Since $\beta_k \not\preceq_r \Delta_T$, either $\sigma_k \not\preceq_r \Delta_T$ or $\alpha_k \not\preceq_r \Delta_T$. Hence either $\eta_{k+1} \not\preceq_r \Delta_T$ or $\eta_k \not\preceq_r \Delta_T$ so $k(bg) = k(b'\gamma_1) \geq k = k(b')$. By the induction hypothesis, $k(b') \geq k(b)$ so $k(bg) \geq k(b)$. $\square$

Let $b = \beta_1 \beta_2 \cdots \beta_k$ (rmd). Then by Lemma 10,

$$\Delta_T^m b = \delta_1 \delta_2 \cdots \delta_\ell \beta_1' \beta_2' \cdots \beta_k' \text{ (rmd)}$$

for some $\beta_1', \beta_2', \ldots, \beta_k' \in M$, and $\delta_1, \delta_2, \ldots, \delta_\ell \preceq_r \Delta_T$. By Lemma 11,

$$b_1 = \Delta_T^m b g^{-1} = \delta_t' \cdots \delta_2' \delta_1' \beta_j'' \cdots \beta_2'' \beta_1'' \text{ (rmd)},$$

where $j \le k$, and $\delta_1', \delta_2', \dots, \delta_t' \preceq_r \Delta_T$. Since n is in the stable range of $h = b_1 \wedge_\ell \Delta_T^n$, $n \ge t$. Thus, $\delta_t' \cdots \delta_2' \delta_1' \preceq_\ell h$ and hence $b_2 = h^{-1} b_1 \preceq_r \beta_j'' \cdots \beta_2'' \beta_1''$. This gives $|b_2|_M \le j \le k = |b|_M$, which is the desired inequality.

Suppose $(X, \preceq_\ell)$ is a poset with least element x_0. If $Y \subseteq X$ and $x_0 \in Y$ then x_0 is the least element of Y. Thus part (iii) is a consequence of part (i) and the following lemma.

Lemma 12. *If $A_U \cap xA_T \ne \emptyset$ then $m(xA_T) \in A_U$.*

Proof. Suppose $y \in A_U \cap xA_T$. Then by Lemma 9, $m(xA_T) = m_T(y)$. Write $y = ab^{-1}$. Then $a, b \in A_U^+$. Using the notation of Eqs. (1)–(6), since $a_1 \preceq_\ell a$, $a_1 \in A_U^+$. Since $m(xA_T) = m_T(y) = a_1 b_2^{-1}$, it suffices to show $b_2 \in A_U^+$. Since $b_1 = \Delta_T^m b g^{-1} \in \Delta_T^m A_U \cap A^+$, the intersection is nonempty. Therefore, there is a least element d_0 with respect to $\preceq_\ell$ of $\Delta_T^m A_U$ (Lemma 8). Thus, $b_1 \in d_0 A_U^+$ so $b_1 = d_0 e$ for some $e \in A_U^+$. Since $d_0 \preceq_\ell \Delta_T^m \preceq_\ell \Delta_T^n$, $h = b_1 \wedge_\ell \Delta_T^n = d_0 e \wedge_\ell d_0(d_0^{-1} \Delta_T^n) = d_0 f$, where $f = e \wedge_\ell d_0^{-1} \Delta_T^n \preceq_\ell e$. Thus, $b_2 = h^{-1} b_1 = f^{-1} d_0^{-1} d_0 e = f^{-1} e \in A_U^+$. □

Proposition 1. *The language L_T of minimal coset representatives of A_T in a finite-type Artin group A is regular.*

We note some facts about regular languages. The details can be found in [9]. Let K and L be regular languages over an alphabet M. Then $M^* - K$, $K \cap L$, $K \cup L$, and the concatenation $KL = \{kl \in M^* : k \in K,\ l \in L\}$ are regular. Let $L_0 = \{\varepsilon\}$, and let $L^n = LL^{n-1}$, for $n = 1, 2, \dots$. The *Kleene closure* of L is $L^* = \bigcup_{n=0}^{\infty} L^n$. It is not hard to see that L^* is regular if L is. Note that L^* contains the empty word even if L does not. Note also that the Kleene closure M^* of the finite language M in the monoid of words M^* generated by the alphabet M is exactly M^* so the notations agree. Define the *reverse* of a word $w = \mu_1, \mu_2, \dots, \mu_n$ to be $\operatorname{rev}(w) = \mu_n, \mu_{n-1}, \dots, \mu_1$, the word spelled backwards. Then the reverse $\operatorname{rev}(L) = \{\operatorname{rev}(w) : w \in L\}$ of a regular language L is regular. Thus if M^{-1} is a set of formal inverses of M, the formal inverse L^{-1} of a regular language L over M is a regular language over M^{-1}. (Replace each label $\mu \in M$ of an arrow of the FSA for $\operatorname{rev}(L)$ with the label μ^{-1} to get an FSA for L^{-1}.)

Let R be the set of words over M representing right greedy minimal decompositions. Let $L_1 = \{ab^{-1} \in RR^{-1} : a \perp_r b\}$. It is shown in [5] that R and L_1 are regular and that in fact, L_1 gives a biautomatic structure with uniqueness for A. Let $L_T = \{w \in L_1 : \overline{w} = m(\overline{w} A_T)\}$. By Theorem 1, L_T is in one-to-one correspondence with the left cosets of A_T in A; i.e., L_T is a normal form with uniqueness for A/A_T.

Proof of Proposition 1. Let $L_2' = \{ab^{-1} \in RR^{-1} : a \perp_r \Delta_T b\}$ and $L_3' = \{ab^{-1} \in RR^{-1} : b \perp_\ell \Delta_T\}$.

Claim. $L_T = L_2' \cap L_3'$.

Proof. Suppose $ab^{-1} \in L_T$. By the uniqueness property of L_1 and Remark 4, $a = a_1$ and $b = b_2$ in the notation of Eqs. (1)–(6). Thus, by Eqs. (1)–(6), $a \perp_r \Delta_T^m b$ hence $a \perp_r \Delta_T b$. Similarly, $b \perp_\ell \Delta_T^n$ hence $b \perp_\ell \Delta_T$. Thus $ab^{-1} \in L_2' \cap L_3'$. For the converse, we need the fact that for any $a, b \in M^*$, if $a \perp_r \Delta_T b$ then $a \perp_r \Delta_T^m b$ for all $m = 1, 2, \ldots$; and if $a \perp_\ell b\Delta_T$ then $a \perp_\ell b\Delta_T^n$ for all $n = 1, 2, \ldots$. This follows from Lemma 2 and induction. Now suppose $ab^{-1} \in L_2' \cap L_3'$. Then applying Eqs. (1)–(5) and using the above fact we find that $a_1 = a$ and $b_2 = b$. Thus, by Remark 4, $ab^{-1} \in L_1$ and by Eq. (6), $m(\overline{ab^{-1}\Delta_T}) = \overline{ab^{-1}}$. Thus $ab^{-1} \in L_T$. □

Let $L_2 = \{ab^{-1} : a, b \in M^*,\ \ a_{|a|} \perp_r \mathrm{maxmin}_r(\Delta_T b)\}$ and $L_3 = \{ab^{-1} : a, b \in M^*, \mathrm{maxmin}_\ell(b) \perp_\ell \Delta_T\}$. For any $a, b \in M^*$, $a \perp_r b$ iff $\mathrm{maxmin}_r(a) \perp_r \mathrm{maxmin}_r(b)$, and $a \perp_\ell b$ iff $\mathrm{maxmin}_\ell(a) \perp_\ell \mathrm{maxmin}_\ell(b)$. If $a \in R$, then $a_{|a|} = \mathrm{maxmin}_r(a)$. Thus, $L_i' = RR^{-1} \cap L_i$, for $i = 2$ or 3. It follows that $L_T = L_1 \cap L_2 \cap L_3$ so it suffices to show that L_2 and L_3 are regular.

For $w \in M^*$, let w_i denote the ith letter in w. Define w_0 to be 1. Then $w_{|w|}$ is the last letter of w if $|w| > 0$ and $w_{|w|} = 1$ if $w = \varepsilon$. For each $\mu \in M \cup \{1\}$, define $E_\mu = \{w \in M^* : w_{|w|} = \mu\}$.

Claim. *E_μ is regular for each $\mu \in M$.*

Proof. Construct an FSA with states $M \cup \{1\}$, with start state 1, μ the only accept state, and for each $v \in M$, an edge labelled v from s to v for each state s. The language of this FSA is E_μ. □

Claim. *For each $v \in M$, define $F_v = \{w \in M^* : v = \mathrm{maxmin}_r(\Delta_T w)\}$. F_v is regular for each $v \in M$.*

Proof. Construct an FSA with states M, start state Δ_T, accept state v, and for each $\zeta, \eta \in M$, an edge labelled η from ζ to $\mathrm{maxmin}_r(\zeta\eta)$. The language of this FSA is F_v. □

Since $L_2 = \bigcup_{\mu \perp_r v} E_\mu F_v^{-1}$, the above claims imply that L_2 is regular.

Let $\mathscr{B}$ be the FSA with states $M \cup \{1\}$, start state 1, accept states $\{1\} \cup \{\mu \in M : \mu \perp_\ell \Delta_T\}$, and for each state s, an edge labelled v from s to $\mathrm{maxmin}_\ell(vs)$. Then the language of $\mathscr{B}$ is $B = \{b \in M^* : \mathrm{maxmin}_\ell(\mathrm{rev}(b)) \perp_\ell \Delta_T\}$. Thus, $\mathrm{rev}\, B = \{\mathrm{rev}(b) \in M^* : \mathrm{maxmin}_\ell(\mathrm{rev}(b)) \perp_\ell \Delta_T\} = \{b \in M^* : \mathrm{maxmin}_\ell(b) \perp_\ell \Delta_T\}$. Therefore, $L_3 = M^*(\mathrm{rev} B)^{-1}$ is regular. This completes the proof of the proposition. □

3. FC-type minimal coset representatives

We now define a normal form for special cosets of those Artin groups which can be built from finite-type Artin groups by amalgamating over special subgroups. As explained below, these are exactly the Artin groups of FC type. The normal form for FC-type Artin groups generalizes the normal form given above for finite-type Artin groups. The restriction to cosets of the trivial subgroup gives a solution to the word problem for Artin groups of FC type.

Let A be an Artin group with canonical generating set S and associated Coxeter group W. The Artin group A is of *FC type* if condition (a) below is satisfied. Let the class of *iterated special amalgams* (ISA) be the smallest class of Artin groups which is closed under free products amalgamated over special subgroups and which contains the finite-type Artin groups. As observed by Davis, the class ISA is exactly the class of FC-type Artin groups.

Proposition 2. *Let A be an Artin group with canonical presentation $\langle S|R\rangle$ and associated Coxeter group W. Then the following are equivalent*:

(a) *If $T\subseteq S$ and every pair of elements of T generates a finite subgroup of W, then T generates a finite subgroup of W.*

(b) *A is an iterated special amalgam (ISA) of finite-type Artin groups.*

Proof. If A satisfies (a), then either A is of finite type, in which case it clearly satisfies (b), or there are distinct elements s and t of S which generate an infinite subgroup of W. Let A_J provisionally denote the subgroup of A generated by $S-J$. Then A is the amalgamated product of $A_{\{s\}}$ and $A_{\{t\}}$ along $A_{\{s,t\}}$. Continue in this way until A has been decomposed into a nested product of finite-type groups. For (b) implies (a), consider that a special amalgam of two groups satisfying (a) will still satisfy (a) since if T is any subset of generators in the amalgam in which every pair generates a finite subgroup of W, then T must be entirely contained in one of the factor groups and so generates a finite subgroup of its Coxeter group which in turn is a subgroup of W. $\square$

3.1. Definition and properties

Theorem 1 generalizes to FC-type Artin groups. The language of special coset representatives is still regular and the normal form of the distinguished coset representative can be calculated from any word representing any element of the coset in quadratic time.

Theorem 2. *Let A be an Artin group of FC type with canonical generating set S, associated Coxeter group W, and set of minimals M. For each $T\subseteq S$, there is a recursive function $m_T: M^{\pm *}\to M^{\pm *}$ such that for every $w\in M^{\pm *}$,*

(i) $\overline{m_T(w)}\in\overline{w}A_T$,

(ii) *for every $v\in M^{\pm *}$, if $\overline{v}\in\overline{w}A_T$ then $m_T(v)=m_T(w)$,*

(iii) *for all $U\subseteq S$, if $A_U\cap\overline{w}A_T\neq\emptyset$ then $m_T(w)\in M_U^{\pm *}$,*

(iv) *$m_T(M^{\pm *})$ is a regular language over $M^{\pm 1}$, and*

(v) *$m_T(w)$ can be computed in $O(|w|_M^2)$ time.*

To prove this theorem, we will use the following facts about amalgamated products (see [15, Theorem 1]). A *transversal* is a set of coset representatives; i.e., if G is a group and H is a subgroup of G, a subset T of G is called a *transversal* of G/H ($H\backslash G$) if for every $x \in G$, there is exactly one $t \in T$ such that $xH = tH$ ($Hx = Ht$).

Theorem 3. *Let $G = G_1 *_H G_2$ and let C_1 and C_2 be transversals containing* 1 *of G_1/H and G_2/H, respectively. For every $x \in G$, there is a unique finite sequence $(x_1, x_2, \ldots, x_n; a)$ in $(C_1 \cup C_2)^* \times H$ such that $x = x_1x_2 \cdots x_n a$ and* (i) *no x_i is trivial and* (ii) *no two consecutive x_i are in the same transversal.*

Let $x_1x_2 \cdots x_n a$ be called the *amalgam normal form* of x with respect to the given amalgamated product decomposition of G and the given transversals. Let n be called the *amalgam length* of x with respect to the amalgamated product $G = G_1 *_H G_2$ and denote it by $|x|_*$.

Corollary 1. *Let $g, c \in G$. Let $g = g_1g_2 \cdots g_n a$ be the amalgam normal form of g and suppose $g_n \in C_1$. Suppose $|c|_* \le 1$ and let $c = c_1 h$ be the normal form of c in the case that $|c|_* = 1$. Then*

$$gc = \begin{cases} g_1g_2 \cdots g_n g_{n+1} b & \text{if } c \in G_2 - H, \\ g_1g_2 \cdots g_{n-1} g_n' a' & \text{if } c \in G_1 - (g_n a)^{-1} H, \\ g_1 g_2 \cdots g_{n-1} h' & \text{if } c \in (g_n a)^{-1} H, \end{cases}$$

where g_{n+1} is the element of C_2 such that $ac_1 = g_{n+1}a''$ for some $a'' \in H$, $b = a''h$, g_n' is the element of C_1 such that $g_n ac = g_n' a'$ for some $a' \in H$, and $h' = g_n ac$.

Suppose we have group presentations $G_1 = \langle S_1 | R_1 \rangle$, $G_2 = \langle S_2 | R_2 \rangle$, and $H = \langle S_{12} | R_1 \cap R_2 \rangle$, where we abbreviate $S_1 \cap S_2 = S_{12}$. Suppose also that for each $i = 1, 2$, we have (i) for each $c \in C_i$ a chosen word $\hat{c} \in S_i^*$ such that $\bar{\hat{c}} = c$ and (ii) an algorithm which accepts any word $u \in S_i^*$ and returns a pair $(\hat{c}, h) \in \widehat{C_i} \times S_{12}^*$ such that $\overline{u} = c\overline{h}$. Then by the above corollary, there is an algorithm to put any word in $(S_1 \cup S_2)^*$ into an amalgam normal form for G. Given $w \in (S_1 \cup S_2)^*$, parse w into subwords $w = w_1w_2 \ldots w_n$ such that for each $i = 1, \ldots, n$, $w_i \in S_1^* \cup S_2^*$ and for each $i = 1, \ldots, n-1$, if $w_i \in S_1^*$ then $w_{i+1} \in S_2^*$ and vice versa. Apply the appropriate algorithm to w_1 to obtain the pair (u_1, v_1) and replace the subword w_1 of w with u_1v_1. If $\overline{u_1} = 1$ then $u_1v_1 \in S_{12}^*$ so $u_1v_1w_2 \in S_1^* \cup S_2^*$ and we can apply the other algorithm to replace $u_1v_1w_2$ with u_2v_2. Otherwise, $v_1w_2 \in S_1^* \cup S_2^*$ so we can apply one of the algorithms to replace v_1w_2 with u_2v_2. We continue in this fashion until after n steps we have an amalgam normal form of length at most n.

Proof of Theorem 2. The proof is by recursion on ISA so we first verify statements (i)–(v) for finite-type Artin groups. By Remark 4, the function m_T defined above can be considered as having range $M^{\pm *}$. Suppose we are given a word $w \in M^{\pm *}$. Since the finite-type groups are biautomatic, we can write w in right normal form (rnf) $w = ab^{-1}$ with a and b in their right greedy canonical decompositions (rmds) in quadratic time [9, Theorem 2.3.10]. We must check that the operations performed on a and b to obtain $m_T(\overline{w})$ can be done algorithmically in time $O((|a|_M + |b|_M)^2)$. Since rnf is a geodesic normal form, an algorithm of complexity $O((|a|_M + |b|_M)^2)$ will be of complexity $O(|w|_M^2)$. We check that the computations in Eqs. (1)–(6) can be done in quadratic time. By Remark 1 and Lemma 7, the g calculation is $O((|a|_M + (|a|_M + |b|_M))^2)$ and the h calculation is $O((2(|a|_M + |b|_M))^2)$. The other calculations are products in which the length of each factor is a linear function of $|a|_M + |b|_M$. Since (M, rnf) is a biautomatic structure, these can be done in $O((|a|_M + |b|_M)^2)$ time. Thus, part (v) holds for the finite-type case and in particular, m_T is a recursive function. Statements (i)–(iv) for the finite-type case now follow from Remark 1, Lemmas 1, 5, and Proposition 1, respectively.

Now suppose that $A = A_1 *_{A_{12}} A_2$, where $A_1 = A_{T_1}$ and $A_2 = A_{T_2}$ are ISA groups and $A_{12} = A_{T_1 \cap T_2}$. Suppose that for $i = 1,2$, for each $T \subseteq T_i$, there is a recursive function $m(\cdot, A_T, A_i) : M_{T_i}^{\pm *} \to M_{T_i}^{\pm *}$ satisfying conditions (i)–(v) wherein S is replaced by T_i. Let $S = T_1 \cup T_2$. We want to construct a recursive function $m_T(\cdot) = m(\cdot, A_T, A)$: $M^{\pm *} \to M^{\pm *}$ satisfying (i)–(v) for each $T \subseteq S$. Let $T \subseteq S$. Let $w \in M^{\pm *}$. The first step in obtaining $m_T(w)$ is to find the amalgam normal form (anf) of $\overline{w}$. Given a word $u \in M_i^{\pm *}$, let $\hat{c} = m(u, A_{12}, A_i)$ and $h = m(u\hat{c}^{-1}, A_\emptyset, A_i)$. By recursion hypotheses (i)–(iii), $\hat{c}$ and h are well-defined and the pair $(\hat{c}, h)$ has the necessary properties to carry out the algorithm for finding anfs. By recursion hypothesis (v), each coset representative can be found in quadratic time. Thus, we have a recursive quadratic time function $a : M^{\pm *} \to M^{\pm *}$ such that for any $v \in M^{\pm *}$, $a(v)$ is the anf of $\overline{v}$. Let $a(w) = w_1 w_2 \ldots w_n a$ (anf). If $n = 0$, define $m_T(w) = m(w, A_{T \cap T_1}, A_1)$. Otherwise, suppose, without loss of generality, that $w_n \in M_1^{\pm *}$. Replace $w_n a$ with $w_n' = m(w_n a, A_{T \cap T_1}, A_1)$. If $m(w_n', A_{12}, A_1) \neq \varepsilon$ or if $n = 1$, the process terminates. Otherwise, replace $w_{n-1} w_n'$ in the resulting word by $w_{n-1}' = m(w_{n-1} w_n', A_{T \cap T_2}, A_2)$ and continue in this fashion until a terminating condition is encountered. The result of this process will be $m_T(w) = w_1 w_2 \cdots w_k'$, where $w_k' = m(w_k', A_{T \cap T_{i(k)}}, A_{i(k)})$ and $w_k' \notin M_{12}^{\pm *}$ if $k \neq 1$. This is achieved after at most one more pass (backward) through w which is quadratic on each subword w_i hence quadratic in the M-length of w.

This gives us a recursive function m_T which satisfies (v). It also satisfies (i) by (i) of the recursion hypothesis. To prove (ii), we first show that $\overline{m_T(w)}$ has minimal amalgam length among elements of $\overline{w}A_T$. Write $m_T(w) = w_1 w_2 \cdots w_k'$ as above. Our claim is certainly true if $w_k' \in M_{12}^{\pm *}$ since the amalgam length is zero in this case. So we suppose this is not the case. Suppose there is $v = v_1 v_2 \cdots v_m$ in amalgam normal form with $\overline{v} \in \overline{w}A_T$ and $m < k$. Then there is $u = u_1 u_2 \cdots u_\ell c$ (anf) such that $\overline{u} \in A_T$ and $\overline{wu} = \overline{v}$. Consider the sequence

$$w, wu_1, \ldots, wu_1 \cdots u_{\ell-1}, wu_1 \cdots u_{\ell-1} u_\ell c$$

and let $x_0, x_1, \ldots, x_\ell$ be the corresponding sequence of amalgam normal forms. Since $m < k$, there must be a first place j in the sequence such that $|x_j|_* < k$. Then by the above corollary, $x_{j-1} = w_1 w_2 \cdots w_{k-1} w_k'' b$ and $\overline{w_k'' b u_j} \in A_{12}$. But $\overline{w_k'' b} = \overline{w_k' u_1 u_2 \cdots u_{j-1}}$ so $\overline{w_k'' b u_j} A_T = \overline{w_k'} A_T$. Thus, $\overline{w_k'} A_T \cap A_{12} \neq \emptyset$ so by condition (iii) of the recursion hypothesis, $w_k' = m(w_k', A_{T \cap T_{i(k)}}, A_{i(k)}) \in M_{12}^{\pm *}$ and this contradicts our assumption.

To prove (ii), let $v, w \in M^{\pm *}$ and suppose $\overline{v} \in \overline{w}A_T$. Then both $m_T(v)$ and $m_T(w)$ have minimal amalgam length in $\overline{w}A_T$ so we can write $v' = m_T(v) = v_1 v_2 \cdots v_{n-1} v_n$ and $w' = m_T(w) = w_1 w_2 \cdots w_{n-1} w_n$ in the normal form described above. Since $\overline{v'} \in \overline{w'}A_T$ (by (i)), there is $x \in M_T^{\pm *}$ in amalgam normal form $x = x_1 x_2 \cdots x_m a$ such that $\overline{v'x} = \overline{w'}$. Consider the sequence of amalgam normal forms $y_0, y_1, \ldots, y_m$ corresponding to the sequence

$$\overline{v'},\ \overline{v'x_1},\ \ldots,\ \overline{v'x_1 \cdots x_{m-1}},\ \overline{v'x} = \overline{w'}.$$

Each element of the sequence is in $\overline{w}A_T$ and so has amalgam length at least n. By the above corollary and induction, for each $j = 0, 1, \ldots, m$,

$$y_j = v_1 v_2 \cdots v_{n-1} v_{n,j} v_{n+1,j} \cdots v_{n+k_j,j} b_j,$$

where $v_{n,j} \in A_{i(n)}$ and $k_0 = k_m = 0$. Thus, $v_i = w_i$ for $i = 1, 2, \ldots, n-1$ and $a(v_n x) = a(w_n) = w_n$. By (ii) of the recursion hypothesis,

$$\begin{aligned} v_n &= m(v_n, A_{T \cap T_{i(n)}}, A_{i(n)}) \\ &= m(a(v_n x), A_{T \cap T_{i(n)}}, A_{i(n)}) \\ &= m(w_n, A_{T \cap T_{i(n)}}, A_{i(n)}) \\ &= w_n. \end{aligned}$$

For $w \in M^{\pm *}$, let $M(w)$ be the smallest subset of M such that $w \in M(w)^{\pm *}$. Part (iii) is proven by showing that the process of putting a word w into normal form does not increase $M(w)$; i.e., $M(m_T(w)) \subseteq M(w)$. This suffices because if $A_U \cap \overline{w}A_T \neq \emptyset$ then there is $u \in M_U^{\pm *}$ such that $\overline{u}A_T = \overline{w}A_T$. It then follows from part (ii) and the above claim that $M(m_T(w)) = M(m_T(u)) \subseteq M(u) \subseteq M_U^{\pm *}$. Let $w \in M^{\pm *}$. The first step in proving the claim is to show that $M(a(w)) \subseteq M(w)$. Parse w into subwords $w = w_1 w_2 \cdots w_n$ such that $M(w_j) \subseteq T_{i(j)}$, where $i(j) \in \{1, 2\}$ for each $j = 1, 2, \ldots, n$. Let $w_1' = m(w_1, A_{12}, A_{i_1})$ and $a_1 = m(w_1'^{-1} w_1, A_\emptyset, A_{i(1)})$. By recursion hypothesis (iii), $M(w_1') \subseteq M(w_1)$ and $M(a_1) \subseteq M(w_1) \cap M_{12}$. Thus, $M(w_1' a_1) \subseteq M(w_1)$ so altering w replacing w_1 with $w_1' a_1$ does not increase $M(w)$. Similarly, at the jth stage of converting w to $a(w)$, we let $w_j' = m(a_{j-1} w_j, A_{12}, A_{i(j)})$ and $a_j = m(w_j'^{-1} a_{j-1} w_j, A_\emptyset, A_{i(j)})$. Then $M(w_j') \subseteq M(a_{j-1} w_j)$ and $M(a_j) \subseteq M(a_{j-1} w_j) \cap M_{12}$. Thus, $M(a(w)) \subseteq M(w)$. The second and final step is to show that if $w = w_1 w_2 \cdots w_n a$ is in anf, then $M(m_T(w)) \subseteq M(w)$. Let $w_n' = m(w_n a, A_{T \cap T_{i(n)}}, A_{i(n)})$. Then $M(w_n') \subseteq M(w_n a)$ and replacing $w_n a$ with w_n' will not increase $M(w)$. If $m(w_n', A_{12}, A_{i(n)}) = \varepsilon$, we replace $w_{n-1} w_n'$ with $w_{n-1}' = m(w_{n-1} w_n', A_{T \cap T_{i(n-1)}}, A_{i(n-1)})$ without increasing $M(w)$.

To prove (iv), let $L_i = m(M^{\pm *}, A_{12}, A_i)$ and $N_i = m(M^{\pm *}, A_{T \cap T_i}, A_i)$. By recursion hypothesis (iv), L_i and N_i are regular. Let $L^- = L_i - \{\varepsilon\}$. Then L_i^- is regular. By the definition of m_T,

$$m_T(M^{\pm *}) = L_1 (L_2^- L_1^-)^* N_2 \cup L_2 (L_1^- L_2^-)^* N_1$$

so $m_T(M^{\pm *})$ is a regular language over $M^{\pm *}$. $\square$

3.2. A^+ injects

Given an Artin group A with canonical presentation, let A^+ denote the monoid with the same presentation. Deligne has shown [8, Theorem 4.14] that the canonical map $\pi : A^+ \to A$ is injective in the case that A is of finite type. This holds for FC-type Artin groups as well.

Theorem 4. *Let A be an Artin group of FC-type. Then the canonical map $\pi : A^+ \to A$ is injective.*

Proof. For two positive words u and v, let $u \sim_{A^+} v$ denote the statement that u and v represent the same element in the monoid A^+. Let $\widetilde{u}$ denote the monoid equivalence class of the word u. Let u and v be positive words such that $\pi(\widetilde{u}) = \pi(\widetilde{v})$ and suppose $x, y \in F(S)^+$ are normal forms such that $u \sim_{A^+} x$ and $v \sim_{A^+} y$. Then $\pi(\widetilde{x}) = \pi(\widetilde{u}) = \pi(\widetilde{v}) = \pi(\widetilde{y})$ and so $x = y$ by the uniqueness property of the normal form. Hence, $u \sim_{A^+} x = y \sim_{A^+} v$. Thus, it suffices to show that any positive word can be brought into normal form by monoid equivalences.

Suppose A is of finite type and $T \subseteq S$. Let $w \in M^*$. Let $w' = m(w, A_T, A)$ and $a = m(w'^{-1}w, A_\emptyset, A)$. Then by Eqs. (1)–(6), $w' \in M^*$ and $a \in M_T^*$. Therefore, by Deligne's theorem, $w \sim_{A^+} w'a$.

Suppose A_1 and A_2 are FC-type Artin groups with the same property; i.e., for $i \in \{1, 2\}$, for all $T \subseteq T_i$, for all $w \in M_i^*$, $w \sim_{A_i^+} w'a$, where $w' = m(w, A_T, A_i)$ and $a = m(w'^{-1}w, A_\emptyset, A_i)$. Suppose $A = A_1 *_{A_{12}} A_2$. Let $T \subseteq S$. Let $w \in M^*$. Parse $w = w_1 w_2 \cdots w_n$ as usual. By the recursion hypothesis, $w_1 \sim_{A_{i(1)}^+} w_1' a_1$, where $w_1' = m(w, A_{12}, A_{i(1)})$ and $a_1 = m(w_1'^{-1} w_1, A_\emptyset, A_{i(1)})$. Similarly, $a_1 w_2 \sim_{A_{i(2)}^+} w_2' a_2$, etc., so we have $w \sim_{A^+} a(w)$ since every equivalence in A_i^+ is an equivalence in A^+. So, without loss of generality, assume $w = a(w) = w_1 w_2 \cdots w_n a$. Let $w_n' = m(w_n a, A_{T \cap T_{i(n)}}, A_{i(n)})$ and $k_n = m(w_n'^{-1} w_n a, A_\emptyset, A_{i(n)})$. If $n \neq 1$ and $m(w_n', A_{12}, A_{i(n)}) = \varepsilon$, let $w_{n-1}' = m(w_{n-1} w_n', A_{T \cap T_{i(n-1)}}, A_{i(n-1)})$ and $k_{n-1} = m(w_{n-1}'^{-1} w_{n-1} w_n', A_\emptyset, A_{i(n-1)})$, etc., according to the algorithm for finding $m_T(w) = w_1 w_2 \cdots w_m'$. Then $w_j' k_j \sim_{A_{i(j)}^+} w_j w_{j+1}'$ for $j = m, \ldots, n-1$ and $w_n' k_n \sim_{A_{i(n)}^+} w_n a$. Let $k = a(k_m k_{m+1} \cdots k_n) = u_1 u_2 \cdots u_\ell b$. If $\ell = 0$, let $k' = m(k, A_\emptyset, A_1)$. Otherwise, let $k' = u_1 u_2 \cdots u_{\ell-1} u_\ell'$, where $u_\ell' = m(u_\ell b, A_\emptyset, A_{i(\ell)})$. Note that $u_\ell' \neq \varepsilon$. Then $w \sim_{A^+} m_T(w) k'$. Therefore, the desired property holds for A as well. Taking $T = \emptyset$ yields that w and its normal form represent the same element of A^+. $\square$

3.3. Asynchronous automaticity

For paths u and $v : [0, \infty) \to X$ in a geodesic metric space X and a constant $k > 0$, we say that u and v are (*synchronous*) *k-fellow travellers* if the uniform distance between u and v is bounded by k; i.e., if $d_X(u(t), v(t)) \leq k$, for all $t \geq 0$. Let $u \overset{k}{\simeq} v$ denote that u and v are k-fellow travellers. We say that u and v are *asynchronous k-fellow travellers*, denoted $u \overset{k}{\sim} v$, if there are unbounded nondecreasing functions $\phi, \psi : [0,\infty) \to [0,\infty)$ such that $u\phi \overset{k}{\simeq} v\psi$.

The definition of an asynchronously automatic group is long and will not be given here. The reader is referred to [2]. The following fact will be used to show that FC-type Artin groups are asynchronously automatic (see [2, Theorem 7.3, Section II]), Let G be a group, M a set of semigroup generators for G, and $\Gamma = \Gamma(G,M)$ the Cayley graph. For a word w over $S \cup S^{-1}$, the convention is to let w also represent the corresponding continuous path $w : [0,\infty) \to \Gamma$ from the identity to $\overline{w}$ which is parameterized by arc length on $[0,|w|]$ and constant on $[|w|,\infty)$. Suppose L is a regular language over M that maps finite-to-one onto G under the canonical map and suppose there is a constant k such that for all $u, v \in L$, $\mu \in M$, if $\overline{u} = \overline{v\mu}$ then $u \overset{k}{\sim} v$. Then (M,L) is an asynchronously automatic structure for G.

Let A be a finite-type Artin group. For $w \in M^{\pm *}$, let $r(w)$ denote the rnf of w. Recall that if w is positive, the rnf is the rmd. It is shown in [5] that $r(M^{\pm *})$ is a biautomatic normal form for A. Let K be a bidirectional fellow travelling constant for this normal form; that is, let K be a positive real number such that for any rnf w and any $\sigma \in M^{\pm 1}$, $\sigma w \overset{K}{\simeq} r(\sigma w)$ and $w\sigma \overset{K}{\simeq} r(w\sigma)$. Then the language of coset representatives for finite-type Artin groups satisfies the following left fellow traveller property.

Lemma 13. *Let A be a finite-type Artin group, $T \subseteq S$, $\sigma \in M$, and $x \in A$ such that $m_T(x) = x$. Then for any $\varepsilon \in \{1,-1\}$, there is $\eta \in M_T \cup \{1\}$ such that* (i) $\overline{\sigma^\varepsilon x} = \overline{m_T(\sigma^\varepsilon x)\eta^\varepsilon}$ *and* (ii) $\sigma^\varepsilon x \overset{2K}{\simeq} m_T(\sigma^\varepsilon x)\eta^\varepsilon$.

Proof. Part (i) is first proven for $\varepsilon = 1$. Let a and b be the rmds in M^* such that ab^{-1} is the rnf of x. Then by Eqs. (1)–(6) with $a = a_1$ and $b = b_2$, $a \perp_r \Delta_T^n b$, for all $n \geq 0$. By Lemma 3, $\sigma a \wedge_r \Delta_T^n b \in M \cup \{1\}$, for all $n \geq 0$. Let J be the stable range of $\sigma a \wedge_r \Delta_T^n b$. Let $N \in J$ and let $g = \sigma a \wedge_r \Delta_T^N b$. Let $\mu = \operatorname{maxmin}_r(\Delta_T^N b)$. Then by Lemma 2, $\mu = \operatorname{maxmin}_r(\Delta_T b)$. Since $g \in M \cup \{1\}$, we have $g \preceq_r \mu$. Thus, $J = \{n \in \mathbf{N} : n \geq 1\}$ and we may take $N = 1$. Also note that $b \wedge_\ell \Delta_T^m = 1$ so $\Delta_T b \wedge_\ell \Delta_T^m = \Delta_T$, for $m \geq 1$. Now find $m_T(\sigma x)$ according to Eqs. (1)–(6) with σa in place of a.

$$
\begin{aligned}
g &= \sigma a \wedge_r \Delta_T b \in M \cup \{1\}, \\
a_1 &= \sigma a g^{-1}, \\
b_1 &= \Delta_T b g^{-1}, \\
h &= b_1 \wedge_\ell \Delta_T^m, \\
b_2 &= h^{-1} b_1, \\
m_T(\sigma a b^{-1}) &= a_1 b_2^{-1}.
\end{aligned}
$$

Thus,

$$m_T(\sigma x) = \sigma a b^{-1} \Delta_T^{-1} h.$$

Since $b_1 \preceq_\ell \Delta_T b$,

$$h = b_1 \wedge_\ell \Delta_T^m \preceq_\ell \Delta_T b \wedge_\ell \Delta_T^m = \Delta_T.$$

Taking $\eta = h^{-1}\Delta_T$ gives part (i) for $\varepsilon = 1$. To show part (i) for $\varepsilon = -1$, let $y = m_T(\sigma^{-1}x)$. By the above argument, there is $\eta \in M \cup \{1\}$ such that $m_T(\sigma y) = \sigma y \eta^{-1}$. Thus,

$$\begin{aligned} m_T(\sigma^{-1}x) &= m_T(y) = y \\ &= \sigma^{-1}\sigma y \eta^{-1}\eta \\ &= \sigma^{-1} m_T(\sigma y)\eta \\ &= \sigma^{-1} m_T(x)\eta. \end{aligned}$$

By the definition of K,

$$\sigma^\varepsilon x \overset{K}{\simeq} r(\sigma^\varepsilon x) = r(m_T(\sigma^\varepsilon x)\eta^\varepsilon) \overset{K}{\simeq} m_T(\sigma^\varepsilon x)\eta^\varepsilon.$$

Thus, part (ii) follows from the triangle inequality and part (i). $\square$

The left fellow travelling property for the language of special cosets extends asynchronously to all Artin groups of FC-type via the following lemma (Fig. 2).

Lemma 14. *Let G be a group with finite generating set S. Let $u = u_1 u_2 \cdots u_n$, $v = v_1 v_2 \cdots v_n$, and $h_0, h_1, \ldots, h_n$ be words over $S \cup S^{-1}$ such that*
(a) $\overline{h_{i-1}u_i} = \overline{v_i h_i}$,
(b) $h_{i-1}u_i \overset{k}{\sim} v_i h_i$, *for $i = 1, \ldots n$, and*
(c) $|h_i| \le \ell$, *for $i = 0, 1, \ldots, n$.*
Then $\overline{h_0 u} = \overline{v h_n}$ and $h_0 u \overset{k+2\ell}{\sim} v h_n$.

Proof. For $i = 1, 2, \ldots, n$, let ϕ_i and ψ_i be the parameterizations such that $h_{i-1}u_i\phi_i \overset{k}{\simeq} v_i h_i \psi_i$. Let $p'(t)$ be the (discontinuous) path that traverses $h_0 u_1$ at speed ϕ_1, jumps to $\overline{v_1}$ and traverses $\overline{v_1}h_1 u_2$ at speed ϕ_2, and so on, ending at $\overline{h_0 u}$ having traversed $\overline{v_1 v_2 \cdots v_{n-1}}h_{n-1}u_n$. See Fig. 2. Similarly, define q' to be the path that traverses $v_1 h_1$, $\overline{v_1}v_2 h_2$, …, $\overline{v_1 v_2 \cdots v_{n-1}}v_n h_n$ in order according to their respective parameterizations ψ_i. Modify p' and q' by having p' wait for q' or vice versa at $\overline{u_1 u_2 \cdots u_i}$ so that both paths jump from that point to $\overline{v_1 v_2 \cdots v_i}$ at the same time for $i = 1, 2, \ldots, n-1$. Let p be the path along $h_0 u$ which coincides with p' except that p waits at $\overline{u_1 u_2 \cdots u_i}$ while p' traverses $\overline{v_1 v_2 \cdots v_i}h_i$. Let q be the path along vh_n which coincides with q' except that q waits at $\overline{v_1 v_2 \cdots v_i}$ while q' traverses $\overline{v_1 v_2 \cdots v_i}h_i$. Then for any $t \ge 0$,

$$\mathrm{d}(p(t), q(t)) \le \mathrm{d}(p(t), p'(t)) + \mathrm{d}(p'(t), q'(t)) + \mathrm{d}(q'(t), q(t)) \le \ell + k + \ell. \qquad \square$$

Lemma 15. *For any FC-type Artin group A, there is a constant $k > 0$ such that for any $T \subseteq S$, any $w \in M^{\pm *}$, and any $\sigma \in M^{\pm 1}$, there is $h \in M_T^{\mp 1} \cup \{1\}$ such that* (i) $\overline{\sigma m_T(w)} = \overline{m_T(\sigma w)h}$ *and* (ii) $\sigma m_T(w) \overset{k}{\sim} m_T(\sigma w)h$.

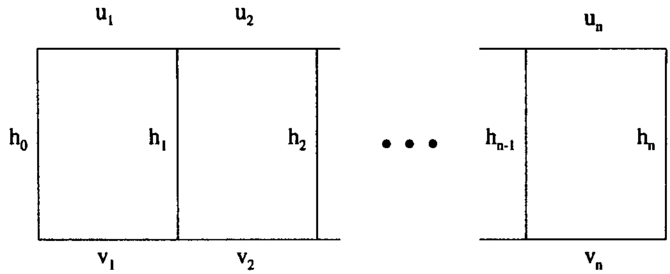

Fig. 2. Concatenating asynchronously fellow travelling paths.

Proof. If A is of finite type, the lemma follows from Lemma 13. Suppose $A = A_1 * A_2$ is a special amalgam of ISA groups and A_1 and A_2 satisfy (i) and (ii). Let k_1 and k_2 be the respective asynchronous fellow travelling constants. Suppose, without loss of generality, that $w = m_T(w)$ and write $w = w_1 w_2 \cdots w_n$ in normal form. We consider separately the cases $\sigma \in M_{i(1)}$ and $\sigma \notin M_{i(1)}$. If $\sigma \in M_{i(1)}$, condition (i) yields

$$
\begin{aligned}
\overline{\sigma w} &= \overline{\sigma w_1 w_2 \cdots w_n}\\
&= \overline{w_1' h_1 w_2 \cdots w_n}\\
&= \overline{w_1' w_2' h_2 \cdots w_n}\\
&= \vdots\\
&= \overline{w_1' w_2' \cdots w_n' h_n},
\end{aligned}
$$

where $h_i \in M_{12}^{\pm 1}$ for $1 \le i < n$ and $h_n \in M_T^{\pm 1}$. Condition (ii) yields $\sigma w_1 \overset{k_{i(1)}}{\sim} w_1' h_1$ in $\Gamma(A_{i(1)}, M_{i(1)})$, and $h_{j-1} w_j \overset{k_{i(j)}}{\sim} w_j' h_j$ in $\Gamma(A_{i(j)}, M_{i(j)})$ for $1 < j \le n$. Since $\Gamma(A_i, M_i)$ may be regarded as a subgraph of $\Gamma(A, M)$, if $u \overset{k}{\sim} v$ in $\Gamma(A_i, M_i)$, then $u \overset{k}{\sim} v$ in $\Gamma(A, M)$. Part (ii) in this case follows from Lemma 14 with $k = \max\{k_1, k_2\} + 2$. If $\sigma \notin M_{i(1)}$, putting σw into normal form yields

$$
\begin{aligned}
\overline{\sigma w} &= \overline{\sigma w_1 w_2 \cdots w_n}\\
&= \overline{w_0' \sigma' w_1 w_2 \cdots w_n}\\
&= \overline{w_0' w_1' h_1 w_2 \cdots w_n}\\
&= \overline{w_0' w_1' w_2' h_2 \cdots w_n}\\
&= \vdots\\
&= \overline{w_0' w_1' w_2' \cdots w_n' h_n}.
\end{aligned}
$$

Since $\sigma \in M^{\pm 1}$, we have $|w_0'|, |\sigma'| \le 1$. Let $w_0 = \varepsilon$. Then $\overline{\sigma w_0} = \overline{w_0' \sigma'}$ and $\sigma w_0 \overset{2}{\sim} w_0' \sigma'$. Thus, Lemma 14 yields (ii) with $k = \max\{k_1, k_2, 2\} + 2$. This choice of k also works in the previous case. $\square$

Theorem 5. *Let A be an FC-type Artin group and let M be the set of minimals of A. Let $L = \mathrm{rev}\{m_\emptyset(w) : w \in M^{\pm *}\}$. Then (M, L) is an asynchronously automatic structure for A.*

Proof. By Theorem 2(iv), L is the reverse of a regular language and is therefore regular. By Theorem 2(i), the natural map $L \to A$ is one-to-one. By Lemma 15(ii), for any $\sigma \in M^{\pm 1}$, $\sigma m_\emptyset(w) \overset{k+1}{\sim} m_\emptyset(\sigma w)$ so $\mathrm{rev}(m_\emptyset(w))\sigma \overset{k+1}{\sim} \mathrm{rev}(m_\emptyset(\sigma w))$. Thus, by the characterization of asynchronously automatic given above, (M, L) is an asynchronously automatic structure for A. $\square$

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